Chase Joyner

801 Homework 6

November 24, 2015

Problem 1:

A scale has two pans. The measurement given by the scale is the difference between the weights in pan #1 and pan #2 plus a random error. Thus, if a weight μ_1 is put in pan #1, a weight μ_2 is put in pan #2, then the measurement is $Y = \mu_1 - \mu_2 + \epsilon$. Suppose that $\mathrm{E}(\epsilon) = 0$, $\mathrm{Var}(\epsilon) = \sigma^2$, and that in repeated uses of the scale, observations Y_i are independent.
Suppose that two objects, #1 and #2, have weights β_1 and β_2. Measurements are then taken as follows:

1. Object #1 is put on pan #1, nothing on pan #2.
2. Object #2 is put on pan #1, nothing on pan #2.
3. Object #1 is put on pan #1, object #2 on pan #2.
4. Objects #1 and #2 are both put on pan #1.

(a) Let $\mathbf{Y} = (Y_1, Y_2, Y_3, Y_4)'$ be the vector of observations. Formulate this as a linear model.

(b) Find vectors $\mathbf{a}_1$ and $\mathbf{a}_2$ such that $\widehat{\beta}_1 = \mathbf{a}_1^T\mathbf{Y}$ and $\widehat{\beta}_2 = \mathbf{a}_2^T\mathbf{Y}$ are the least squares estimators of β_1 and β_2.

(c) Find the covariance matrix for $\widehat{\beta} = (\widehat{\beta}_1, \widehat{\beta}_2)'$.

(d) Find a matrix $\mathbf{A}$ such that $S^2 = \mathbf{Y}'\mathbf{A}\mathbf{Y}$.

(e) For the observation $\mathbf{Y} = (7, 3, 1, 7)'$, find S^2, and estimate the covariance matrix of $\widehat{\beta}$.

(f) Show that four such weighings can be made in such a way that the least squares estimators of β_1 and β_2 have smaller variances than for the experiment above.

Solution:

(a) The linear model for this experiment can be written as

$$Y = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & -1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} \beta_1 \\ \beta_2 \end{bmatrix} + \begin{bmatrix} \epsilon_1 \\ \epsilon_2 \\ \epsilon_3 \\ \epsilon_4 \end{bmatrix}.$$

(b) Recall that the least squares estimator for β is $\widehat{\beta} = (X'X)^{-}X'Y$. So, it follows that

$$a = (X'X)^{-}X' = \begin{bmatrix} \frac{1}{3} & 0 & \frac{1}{3} & \frac{1}{3} \\ 0 & \frac{1}{3} & -\frac{1}{3} & \frac{1}{3} \end{bmatrix}.$$

Therefore, $a_1' = (1/3, 0, 1/3, 1/3)$ and $a_2' = (0, 1/3, -1/3, 1/3)$.

(c) By part (b), we have $\widehat{\beta} = a^T Y$. Therefore, we have

$$\mathrm{Cov}(\widehat{\beta}) = \mathrm{Cov}(a^T Y) = a^T \mathrm{Cov}(\mathrm{Y}) a = \sigma^2 a^T a = \sigma^2 (X'X)^{-} = \sigma^2 \begin{bmatrix} \frac{1}{3} & 0 \\ 0 & \frac{1}{3} \end{bmatrix}.$$

(d) Recall that an estimate of σ^2 is

$$S^2 = MSE = \frac{Y'(I-M)Y}{r(I-M)}.$$

It follows that A is the matrix

$$A = \frac{I-M}{r(I-M)}.$$

Note that

$$M = X(X'X)^{-}X' = \begin{bmatrix} \frac{1}{3} & 0 & \frac{1}{3} & \frac{1}{3} \\ 0 & \frac{1}{3} & -\frac{1}{3} & \frac{1}{3} \\ \frac{1}{3} & -\frac{1}{3} & \frac{2}{3} & 0 \\ \frac{1}{3} & \frac{1}{3} & 0 & \frac{2}{3} \end{bmatrix}.$$

Therefore, we have

$$I - M = \begin{bmatrix} \frac{2}{3} & 0 & -\frac{1}{3} & -\frac{1}{3} \\ 0 & \frac{2}{3} & \frac{1}{3} & -\frac{1}{3} \\ -\frac{1}{3} & \frac{1}{3} & \frac{1}{3} & 0 \\ -\frac{1}{3} & -\frac{1}{3} & 0 & \frac{1}{3} \end{bmatrix}.$$

Then, notice $r(I-M) = r(I) - r(M) = 4 - 2 = 2$, and so

$$A = \frac{I-M}{r(I-M)} = \begin{bmatrix} \frac{1}{3} & 0 & -\frac{1}{6} & -\frac{1}{6} \\ 0 & \frac{1}{3} & \frac{1}{6} & -\frac{1}{6} \\ -\frac{1}{6} & \frac{1}{6} & \frac{1}{6} & 0 \\ -\frac{1}{6} & -\frac{1}{6} & 0 & \frac{1}{6} \end{bmatrix}.$$

(e) By part (d), we have

$$S^2 = Y'AY = \begin{bmatrix} 7 & 3 & 1 & 7 \end{bmatrix} \begin{bmatrix} \frac{1}{3} & 0 & -\frac{1}{6} & -\frac{1}{6} \\ 0 & \frac{1}{3} & \frac{1}{6} & -\frac{1}{6} \\ -\frac{1}{6} & \frac{1}{6} & \frac{1}{6} & 0 \\ -\frac{1}{6} & -\frac{1}{6} & 0 & \frac{1}{6} \end{bmatrix} \begin{bmatrix} 7 \\ 3 \\ 1 \\ 7 \end{bmatrix} = 3.$$

Also, by part (b), an estimate of $\text{Cov}(\widehat{\beta})$ is

$$\widehat{\text{Cov}(\widehat{\beta})} = \widehat{\sigma}^2 \begin{bmatrix} \frac{1}{3} & 0 \\ 0 & \frac{1}{3} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

(f) We can change the experiment to give design matrix

$$X = \begin{bmatrix} 1 & -1 \\ 1 & 1 \\ 1 & -1 \\ 1 & 1 \end{bmatrix}.$$

Then, we calculate

$$\text{Cov}(\widehat{\beta}) = \sigma^2 (X'X)^{-} = \sigma^2 \begin{bmatrix} \frac{1}{4} & 0 \\ 0 & \frac{1}{4} \end{bmatrix}.$$

Since the diagonal of this matrix is smaller than before, we conclude this experiment gives smaller variances for the LSE of β_1 and β_2.

Problem 2:

Consider the weighing problem above with the two unknown weights β_1 and β_2.

(a) Suppose we want Scheffe 95% confidence intervals on all linear combinations of β_1 and β_2. For the four weighings made and for $\mathbf{Y} = (7, 3, 1, 7)'$, find these intervals for the three linear combinations β_1, β_2, and $\beta_1 - \beta_2$.

(b) Use the Bonferroni method to find 95% simultaneous confidence intervals on these same three linear combinations.

(c) Suppose that we wish to test $H_0\colon \beta_1 = \beta_2 = 0$. Then, $\sup_{\mathbf{c} \in C} t_c^2/q = F$ is the corresponding F–statistic. For which value of $\mathbf{c}$ does $t_c^2/q = F$? What is the corresponding $\mathbf{a}_c$?

(d) Find a 95% confidence ellipsoid on $\boldsymbol{\beta} = (\beta_1, \beta_2)'$ for these data.

Solution:

(a) Recall that a $100(1-\alpha)\%$ Scheffe confidence intervals for all linear functions of β are given by

$$\left[c^T \widehat{\beta} \pm \sqrt{p \cdot MSE \cdot f_{\alpha,p,n-p} \cdot c^T (X^T X)^{-1} c} \right].$$

We found $MSE = 3$ in part (e) of problem 1. Note that the design matrix is

$$X = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & -1 \\ 1 & 1 \end{bmatrix}$$

and so $p = r(X) = 2$. If $c_1 = (1, 0)'$, $c_2 = (0, 1)'$, and $c_3 = (1, -1)'$, then

$$c_1'\beta = \beta_1 \quad c_2'\beta = \beta_2 \quad c_3'\beta = \beta_1 - \beta_2.$$

Lastly, by part (a) from problem 1,

$$\widehat{\beta} = \begin{bmatrix} \frac{1}{3} & 0 & \frac{1}{3} & \frac{1}{3} \\ 0 & \frac{1}{3} & -\frac{1}{3} & \frac{1}{3} \end{bmatrix} Y = \begin{bmatrix} 5 \\ 3 \end{bmatrix}.$$

Therefore, the intervals for the three linear combinations are

$$\left[c_1'\widehat{\beta} \pm \sqrt{2 \cdot 3 \cdot f_{0.05,2,4-2} \cdot c_1'(X'X)^{-1}c_1}\right] = [-1.164, 11.164],$$
$$\left[c_2'\widehat{\beta} \pm \sqrt{2 \cdot 3 \cdot f_{0.05,2,4-2} \cdot c_2'(X'X)^{-1}c_2}\right] = [-3.164, 9.164],$$
$$\left[c_3'\widehat{\beta} \pm \sqrt{2 \cdot 3 \cdot f_{0.05,2,4-2} \cdot c_3'(X'X)^{-1}c_3}\right] = [-6.718, 10.718].$$

(b) Recall that a $100(1 - \alpha_i)\%$ Bonferroni confidence interval is

$$\left[c_i'\widehat{\beta} \pm t_{\alpha_i/2,n-p}\sqrt{d_i \cdot MSE}\right],$$

where $d_i = c_i'(X'X)^{-1}c_i$. Note that $MSE = 3$. Let $\alpha_1 = \alpha_2 = \alpha_3 = \alpha/3$. Using the same c_1, c_2, and c_3 as in part (a), we have

$$d_1 = c_1'(X'X)^{-1}c_1 = \frac{1}{3}$$
$$d_2 = c_2'(X'X)^{-1}c_2 = \frac{1}{3}$$
$$d_3 = c_3'(X'X)^{-1}c_3 = \frac{2}{3}.$$

Then, the Bonferroni confidence intervals for the three linear combinations are

$$\left[c_1'\widehat{\beta} \pm t_{\alpha_1/2,4-2}\sqrt{d_1 \cdot MSE}\right] = [-2.649, 12.649]$$
$$\left[c_2'\widehat{\beta} \pm t_{\alpha_2/2,4-2}\sqrt{d_2 \cdot MSE}\right] = [-4.649, 10.649]$$
$$\left[c_3'\widehat{\beta} \pm t_{\alpha_3/2,4-2}\sqrt{d_3 \cdot MSE}\right] = [-8.817, 12.817].$$

(c) Note that under H_0, $X_0 = 0_{4\times 2}$ and $M_0 = 0_{4\times 4}$. First, recall

$$F = \frac{Y'(M - M_0)Y/r(M - M_0)}{Y'(I - M)Y/r(I - M)} = \frac{Y'MY/r(M)}{MSE}.$$

We see that

$$Y'MY = \begin{bmatrix} 7 & 3 & 1 & 7 \end{bmatrix} \begin{bmatrix} \frac{1}{3} & 0 & \frac{1}{3} & \frac{1}{3} \\ 0 & \frac{1}{3} & -\frac{1}{3} & \frac{1}{3} \\ 0 & -\frac{1}{3} & \frac{2}{3} & 0 \\ \frac{1}{3} & \frac{1}{3} & 0 & \frac{2}{3} \end{bmatrix} \begin{bmatrix} 7 \\ 3 \\ 1 \\ 7 \end{bmatrix} = 102.$$

Also, $r(M) = 2$ and $MSE = 3$. Therefore, $F = 17$. Again, $r(X) = q = 2$ and so we want to find a c such that $t_c^2 = 34$. Note that

$$t_c = \frac{c'(\widehat{\beta} - \beta)}{\sqrt{MSE \cdot c'(X'X)^{-1}c}},$$

and since $\beta = (0,0)'$, $MSE = 3$, and squaring both sides, we have

$$t_c^2 = \frac{(5c_1 + 3c_2)^2}{c_1^2 + c_2^2}.$$

Setting this equal to 34 and solving for c_1 and c_2, we have $c = (15, 9)'$ is a solution. Then, the corresponding a_c is

$$a_c = X(X'X)^{-1}c = \begin{bmatrix} \frac{1}{3} & 0 \\ 0 & \frac{1}{3} \\ \frac{1}{3} & -\frac{1}{3} \\ \frac{1}{3} & \frac{1}{3} \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 5 \\ 3 \\ 2 \\ 8 \end{bmatrix}.$$

(d) Recall that from our notes

$$\frac{W^\star}{q} \sim F(q, n-p),$$

where $W^\star = \sup_{c \in C} t_c^2 = \frac{(\widehat{\beta}-\beta)'(X'X)(\widehat{\beta}-\beta)}{S_c^2}$. From this, we see that a 95% confidence ellipsoid on β is

$$(\widehat{\beta} - \beta)'(X'X)(\widehat{\beta} - \beta) \le S_c^2 \cdot q \cdot f(\alpha, q, n-p).$$

We calculate

$$(\widehat{\beta} - \beta)'(X'X)(\widehat{\beta} - \beta) = \left(\begin{bmatrix} 5 \\ 3 \end{bmatrix} - \begin{bmatrix} \beta_1 \\ \beta_2 \end{bmatrix} \right)' \begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix} \left(\begin{bmatrix} 5 \\ 3 \end{bmatrix} - \begin{bmatrix} \beta_1 \\ \beta_2 \end{bmatrix} \right) \le 3 \cdot 2 \cdot 19$$

and so we see that

$$(5 - \beta_1)^2 + (3 - \beta_2)^2 \le 38.$$

Therefore, a 95% confidence ellipsoid on β is given by

$$\{\beta \mid (5 - \beta_1)^2 + (3 - \beta_2)^2 \le 38\}.$$